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# Finitely Presented Modules over Semiperfect Rings Satisfying ACC-$\infty$

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Endomorphism rings of finitely presented modules over semiperfect rings are studied, leading to a Krull–Schmidt decomposition theory of modules for finitely generated modules over complete semiperfect Noetherian rings. An example is given of a local non-Noetherian ring having a finitely presented module with pathelogical properties. © 1987 Academic Press, Inc.

$R$ denotes a ring with 1. Recall $R$ is *semiperfect* if $R/\mathrm{Jac}(R)$ is semisimple Artinian and $\mathrm{Jac}(R)$ is idempotent-lifting, cf. [1, p. 303 ff.] which is used as a general reference. For example, every local ring is semiperfect (where for us a lcoal ring need not be commutative nor Noetherian). Conversely every semiperfect ring is Morita equivalent to its "basic" ring which, modulo the Jacobson radical, is a finite direct product of division rings. Consequently the module theory of semiperfect rings might be expected to parallel that of local rings.

By *Krull–Schmidt* theory we mean a theory in which a given module is a finite direct sum of indecomposable submodules whose endomorphism rings are local; equivalently the endomorphism ring of the module is semiperfect, by [1, Theorem 27.6(b)]. In particular if $R$ itself is to have a Krull–Schmidt theory then $R \approx \mathrm{End}_R R$ must be semiperfect. For convenience write f.g. for "finitely generated $R$-module." A module of the form $R^{(n)}/K$ is called *finitely presented* if $K$ is f.g. If $R$ is left Noetherian then every f.g. module is finitely presented.

In [9] it was seen that a Krull–Schmidt theory can be achieved via Fitting's lemma for arbitrary finitely presented modules over a semiperfect ring $R$, iff $R$ is $\pi_\infty$-regular, a condition slightly stronger than the Jacobson radical being nil. However there are important cases of semiperfect rings for which the Jacobson radical is *not* nil, for example any localization of $\mathbb{Z}$. Swan [12] has already proved and applied a Krull–Schmidt theory for algebras finitely spanned over a complete local Noetherian domain;



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Auslander [2] developed a representation theory (generalizing his work on Artin algebras) for a certain class of these algebras.

This paper was motivated by an attempt to generalize these classes of rings and thereby simplify their hypotheses. In other words we are led to try to develop a reasonably general Krull–Schmidt theory for modules over arbitrary noncommutative Noetherian semiperfect rings, which includes these examples.

Unfortunately there is an example of Swan (given in [4]) of a f.g. module over a commutative, Noetherian semiperfect ring, which fails to satisfy a Krull–Schmidt theory. Thus for arbitrary Noetherian semiperfect rings we shall have to be satisfied with a weaker property of the endomorphism ring, that onto endomorphisms are 1 : 1, along the lines of [10]. It also turns out that the Noetherian property can be weakened to the following condition:

DEFINITION 1. A module $M$ *satisfies* ACC-$t$ if there is no infinite chain of submodules $N_1 < N_2 < \cdots$ each spanned by $\leqslant t$ elements. $R$ *satisfies* ACC-$\infty$ if the free $R$-module $R^{(t)}$ satisfies ACC-$t$ for each $t$.

Of course any left Noetherian ring satisfies ACC-$\infty$. However the class of rings satisfying ACC-$\infty$ is considerably more general. Left perfect rings also satisfy ACC-$\infty$ by a theorem of Jonah [6]; other interesting examples can be found in [8, 11]. Let us start with an example indicating why we must restrict our attention to rings with ACC-$\infty$.

EXAMPLE 2. A commutative local (non-Noetherian) ring which does not satisfy ACC-1, having a finitely presented module $M$ over which there is an onto endomorphism which is *not* 1 : 1. This example will be described as part of a general framework which has several applications. Let $R_0$ be any semiperfect ring, and let $\mathscr{F}$ be an ultrafilter of $\mathbb{N}$ (the set of natural numbers) containing the cofinite filter. Taking each $R_i$ to be a copy of $R_0$, let $R$ be the ultraproduct $(\prod_{i \in \mathbb{N}} R_i)/\mathscr{F}$. Recall this is the set of $(r_i)$ in $\prod R_i$, modulo the relation $(r_i) = (r'_i)$ iff $\{i : r_i = r'_i\} \in \mathscr{F}$; $R$ satisfies every elementary sentence satisfied by each $R_i$, cf. [7].

*Claim* 1. Let $J_i = \mathrm{Jac}(R_i)$ and $J = (\prod J_i)/\mathscr{F}$, viewed in $R$. Then $J = \mathrm{Jac}(R)$. (Indeed each element of $J$ is quasi-invertible so $J \subseteq \mathrm{Jac}(R)$. On the other hand if $(r_i) \notin J$ then there is some $I \in \mathscr{F}$ such that for each $i$ in $I$ we can find $a_i$ in $R_i$ such that $1 - a_i r_i$ is not invertible; for each other $i$ take $a_i = 0$. Then $1 - (a_i)(r_i)$ is not invertible in $R$ so $(r_i) \notin \mathrm{Jac}(R)$, implying $J = \mathrm{Jac}(R)$.)

*Claim* 2. If $R_i/J_i \approx M_n(D_i)$ then $R/J \approx (\prod_{i \in I} M_n(D_i))/\mathscr{F} \approx M_n(\prod D_i/\mathscr{F})$. (Indeed the canonical homomorphism $R \to \prod M_n(D_i)/\mathscr{F}$

given by $(r_i) \to (r_i + J_i)$ clearly has kernel $J$, yielding the first isomorphism, and the second isomorphism follows when we note that the matric units of the $R_i$ "line up" to form matric units of $R$.) In particular, taking $n = 1$, we see that if $R_0$ is local then $R$ is local.

(Claims 3 and 4 are not needed here but reflect the spirit of the example.)

*Claim* 3. If $R_0$ is semilocal then $R$ is semilocal. (Indeed Claim 2 is the corresponding assertion for "quasilocal," and the proof for "semilocal" is analogous.)

*Claim* 4. If $R_0$ is semiperfect then $R$ is semiperfect. (Indeed all that remains to show is that idempotents lift modulo $J$. But if $(r_i + J_i)$ is an idempotent in $R/J$ then $\{i: r_i + J_i$ is idempotent in $R_i/J_i\} \in \mathscr{F}$. Each of these $r_i + J_i$ can be rewritten as $e_i + J_i$ for suitable idempotents $e_i$ in $R_i$; taking all other $e_i$ arbitrarily we see $(r_i + J_i) = (e_i + J_i)$ and $(e_i)$ is idempotent in $R$.)

*Claim* 5. Suppose $R_0$ is not right perfect, i.e., $R_0$ fails to satisfy the DCC on principal left ideals. Then $R$ fails ACC-1. (Indeed if $L_1 > L_2 > \cdots$ are principal left ideals of $R_0$ take $L_u = R_0$ for all $u \leqslant 0$ and define $I_k = (\prod_i L_{i-k})/\mathscr{F}$ for each $k > 0$. Writing $L_i = R_i a_i$ we see $I_k = R(a_{i-k})$ also is principal. Clearly $I_k \subset I_{k+1}$ for each $k$ since $L_{i-k} \subset L_{i-k-1}$ for every $i > k+1$, thereby proving $R$ fails ACC-1.)

*Claim* 6. Continuing Claim 5, let $M = R/I_0$, where $I_0 = (\prod L_i)/\mathscr{F}$. $I_0$ is principal so $M$ is finitely presented. On the other hand, there is an onto map $f: M \to M$ which is not $1:1$, defined by $f((r_i + L_i)) = (r_{i+1} + L_i)$ (for if $r_i \in L_{i-1} - L_i$ then $(r_i + L_i) \neq 0$ but $f((r_i + L_i)) = 0$.)

Claims 5 and 6 yield the properties promised at the beginning of the discussion of this example. We take $R_0$ to be a local Noetherian integral domain which is not Artinian; then $R_0$ is not perfect, so $R$ is a local integral domain failing ACC-1. These two claims rest on the same idea, so one could hope that a positive result holds for rings which do satisfy ACC-$\infty$.

PROPOSITION 3. *Suppose $M$ is a finitely presented module over a semiperfect ring $R$ satisfying ACC-$\infty$. Then every onto map $f: M \to M$ is an isomorphism.*

*Proof.* By [1, Theorem 27.6] there is a projective cover $\pi: P \to M$ and $P$ is f.g. On the other hand, we are given an epic $\psi$ from a free module to $M$, having f.g. kernel. Thus ker $\pi$ is f.g. (being a direct summand of ker $\psi$) by [1, Lemma 17.17], which also shows that the kernel of every projective cover of $M$ is spanned by the same number of elements. Hence by hypothesis we can choose $\pi$ with ker $\pi$ maximal. Since $P$ is projective there is a map $g$ completing the following commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f\pi"] & \\ P \arrow[r, "\pi"'] & M \arrow[r] & 0 \end{tikzcd}$$

Then $\pi g = f\pi$. Hence $\pi g P = M = \pi P$ implying $P = gP + \ker \pi$, so $P = gP$ since $\ker \pi$ is small. But $g$ is onto and $P$ is projective, so it follows by a standard argument that $g$ is an isomorphism (cf. [10; or 9, end of proof of Proposition 6]).

On the other hand $\pi g: P \to M$ is a projective cover with $\ker \pi g = \ker f\pi \supseteq \ker \pi$; by choice of $\pi$ we get $\ker \pi g = \ker \pi$. Therefore $\ker f = 0$ (for if $x \in P$ and $0 = f\pi x = \pi g x$ then $\pi x = 0$). Q.E.D.

COROLLARY 4. *Suppose $R$ is semiperfect with ACC-$\infty$, and $M$ is a finitely presented $R$-module with $f_1, f_2 \in \mathrm{End}_R\, M$. If $f_1 f_2$ is onto then $f_1$ and $f_2$ are invertible.*

*Proof.* If $f_1$ were not invertible then $f_1$ would not be onto by Proposition 3, contrary to $f_1 f_2$ onto. Hence $f_1$ is invertible, implying $f_2$ is onto and thus invertible. (Indeed if $x \notin f_2 M$ then $f_1 x \notin f_1 f_2 M$, contrary to $f_1 f_2$ onto.) Q.E.D.

We should like to utilize these results to obtain a Krull–Schmidt theory analogous to [9, Theorem 8]. Unfortunately there is Swan's counterexample mentioned earlier. Nevertheless, positive results including a generalization of Fitting's lemma are available for certain semiperfect rings.

We shall need to consider $f$ and $1-f$ simultaneously. To this end we consider the class of polynomials $\mathscr{P} = \{p \in \mathbb{Z}[\lambda] : p(0) = 0 \text{ and } p(1) = 1\}$. Note that $\mathscr{P}$ satisfies the following properties:

(i) $\mathscr{P}$ is a monoid (i.e., if $p_1(\lambda), p_2(\lambda) \in \mathscr{P}$ then $p_1(\lambda)\, p_2(\lambda) \in \mathscr{P}$);

(ii) If $p_1(\lambda), p_2(\lambda) \in \mathscr{P}$ then $p_1(p_2(\lambda)) \in \mathscr{P}$;

(iii) $\lambda$ divides $p$ and $1-\lambda$ divides $1-p$, for each $p$ in $\mathscr{P}$. (Indeed $(1-p)(1) = 0$.)

(iv) If $p_1(\lambda), p_2(\lambda) \in \mathscr{P}$ then $1-(1-p_1(\lambda)) \cdot (1-p_2(\lambda)) \in \mathscr{P}$; in particular if $p = p_1 = p_2 \in \mathscr{P}$ then $2p - p^2 \in \mathscr{P}$.

Given $p$ in $\mathbb{Z}[\lambda]$ write $p^*$ for $1-p$. Then $p^*(f) = 1 - p(f)$. Since $(1-\lambda) \mid p^*$ by (iii) we could write $p^* = (1-\lambda) \cdot q$ for suitable $q$ in $\mathbb{Z}[\lambda]$.

Now we can improve [9, Theorem 8] as follows:

PROPOSITION 5 (Fitting's lemma modified). *Suppose $R$ is a right perfect* *ring. For any finitely presented module $M$ and $f$ in $\mathrm{End}_R(M)$ one can find suitable* $p_0(\lambda)$ in $\mathscr{P}$ such that putting $f_0 = p_0(f)$ we have

$$M \approx p(f_0)\, M \oplus \ker p(f_0) \approx p^*(f_0)\, M \oplus \ker p^*(f_0) \qquad \text{for all} \quad p(\lambda) \text{ in } \mathscr{P}.$$

*Proof.* Suppose $M$ is spanned by $t$ elements. Since any homomorphic image of $M$ is also spanned by $\leqslant t$ elements we can take $p_1$ in $\mathscr{P}$ with $p_1(f)\, M$ minimal possible. Taking $f_1 = p_1(f)$ we have $p(f_1)^2\, M = p(f_1)\, M$ for all $p$ in $\mathscr{P}$ so by [9, Theorem 8] we have

$$M \approx p(f_1)\, M \oplus \ker p(f_1).$$

Now take $p_2$ in $\mathscr{P}$ with $p_2^*\,(f_1)\, M$ minimal possible. Letting $f_0 = p_2(f_1)$ we have as before $M \approx p(f_0)\, M \oplus \ker p(f_0)$ for all $p(\lambda)$ in $\mathscr{P}$. Furthermore writing $p^* = (1-\lambda)\cdot q$ we have $p^*(f_0)\, M = ((1-\lambda)\cdot q) f_0 M = ((1-f_0)\cdot q(f_0))\, M \subseteq (1-f_0)\, M = p_2^*(f_1)\, M$. But $p^*(f_0) = 1 - p(f_0) = (1-pp_2) f_1 = (pp_2)^* f_1$ so by hypothesis on $p_2$ we see $p^*(f_0)\, M = p_2^*(f_1)\, M$ for all $p$ in $\mathscr{P}$.

Now $p^*(f_0)^2 = (1-p(f_0))^2 = 1 - 2p(f_0) + p(f_0)^2 = (2p-p^2)^* f_0$; applying (iv) above to the preceding paragraph we see $p^*(f_0)^2\, M = p_2^*(f_1)\, M = p^*(f_0)\, M$, implying $M \approx (p^* f_0)\, M \oplus \ker(p^* f_0)$, as desired. Q.E.D.

Let $J = \mathrm{Jac}(R)$. We shall say a ring $R$ *satisfies Jacobson's conjecture* if $\bigcap_{j \in \mathbb{N}} J^j = 0$. Although this assumption is false for semiperfect left Noetherian rings (cf. [5]) it does hold for many left and right Noetherian rings, including left and right Noetherian rings satisfying a polynomial identity, by [3, Sect. 7]. On the other hand, one can construct a commutative local non-Noetherian domain $R$ for which Jacobson's conjecture fails. Indeed apply the construction of Example 2 to a commutative local domain $R_0$ whose Jacobson radical $J_0$ is not nilpotent; if $r_m \in J^m - J^{m+1}$ then $0 \neq (r_m) \in J^i$ for all $i$. (As shown above, this example also fails ACC-$\infty$.)

We say $R$ is *complete* if $R$ satisfies Jacobson's conjecture and is complete with respect to the $\mathrm{Jac}(R)$-adic topology. Modules over complete local (commutative) Noetherian rings were treated in [12], so it makes sense to consider complete semiperfect rings.

The next proposition is needed in the proof of Lemma 6, following easily from a result of Hinohara [13, Lemma 3]:

Hinohara's proposition: Suppose $R$ is complete semilocal, $J$ is f.g. as left $R$-module, and $M = P/K$, where $K$ is a f.g. submodule of a f.g. projective module $P$. Then $K$ is closed in $P$ with respect to the $J$-adic topology of $P$, and $\bigcap_{i \in \mathbb{N}} J^i M = 0$.

PROPOSITION A. *If $R$ is complete semiperfect, $J$ is f.g. as left $R$-module, and $M$ is a finitely presented $R$-module then* $\mathrm{End}_R\, M$ *satisfies Jacobson's conjecture and is complete.*

*Proof.* As in the proof of Proposition 3, we take a projective cover $\pi: P \to M$, where $P$ and $K = \ker \pi$ are f.g. (and thus complete by Jacobson ("Basic Algebra II," Proposition 7.29) whose proof works for any ring). Let $E_0 = \mathrm{End}_R P$ and $E = \{g \in E: gK \subseteq K\}$. Write $\bar{E} = \mathrm{End}_R M$. There is a homomorphism $\varphi: E \to \bar{E}$ given by $\varphi g(\pi x) = \pi(gx)$. As in the proof of Proposition 3 we see $\varphi$ is onto. Consequently $\bar{E} \approx E/A$ where $A = \{g \in \mathrm{End}_R P: gP \subseteq K\}$. Suppose $\{f_i\}$ is a Cauchy sequence in $\bar{E}$, and take $g_i$ in $E$ such that $f_i = \varphi g_i$. By [1, Corollary 17.12] $A \subseteq \mathrm{Jac}(E_0)$. Thus we can pick the $g_i$ inductively on $i$ such that $\{g_i\}$ is a Cauchy sequence in $E_0$. Let $g$ be their limit in $\mathrm{End}_R P$. For any $x$ in $P$, if $g_i - g_j \in \mathrm{Jac}(E_0)^m$ then using [1, Proposition 17.10] we have $g_i(x) - g_j(x) \in J^m P \cap K = J^m K$ using Hinohara's lemma; consequently $\{g_i(x)\}$ is a Cauchy sequence in $K$, so $g(x) \in K$; thus $g \in E$, and $\varphi g$ is the limit of the $f_i$, proving that $\bar{E}$ is indeed complete.

Likewise to see $\bar{E}$ satisfies Jacobson's conjecture suppose $f \in \bigcap \mathrm{Jac}(\bar{E})^i$. As above we could lift $\{f, f, f, f, \ldots\}$ to a Cauchy series in $E$ which converges to 0, so its image 0 is the limit of $\{f, f, \ldots,\}$ which implies $f = 0$. Q.E.D.

Now we have the following general version of Fitting's lemma:

LEMMA 6. *Suppose $R$ is a complete semilocal ring and $J$ is f.g. as $R$-module. Then for any finitely presented $R$-module $M$ there is a suitable sequence of polynomials $p_1, p_2, \ldots,$ in $\mathscr{P}$ such that putting $f_0 = f$ and $f_i = p_i(f_{i-1})$ we have*

$$M \approx \bigcap_{i \in \mathbb{N}} (f_i M + J^i M) \oplus \bigcap_{i \in \mathbb{N}} K_i$$
$$\approx \bigcap_{i \in \mathbb{N}} ((1 - f_i) M + J^i M) \oplus \bigcap_{i \in \mathbb{N}} K'_i$$

*where $K_i = \{x \in M: f_i x \in J^i M\}$ and $K'_i = \{x \in M: (1 - f_i) x \in J^i M\}$.*

*Proof.* Appealing to symmetry, we prove only the first isomorphism. Suppose $M$ is generated by $t$ elements. We work inductively on $i \geqslant 1$, looking in $\bar{R} = R/J^i$, a semilocal ring whose Jacobson radical $\bar{J}$ is nilpotent; hence $\bar{R}$ is semiprimary. Let $\bar{M} = M/J^i M$, viewed naturally as a $\bar{R}$-module. Then $f_{i-1}$ induces a map $\bar{f}_{i-1}: \bar{M} \to \bar{M}$. By Proposition 5 there is some $p_i$ in $\mathscr{P}$ such that putting $f_i = p_i(f_{i-1})$ we have

$$\bar{M} \approx p(\bar{f}_i)\, \bar{M} \oplus \ker p(\bar{f}_i) \approx (1 - p(\bar{f}_i))\, \bar{M} \oplus \ker(1 - p(\bar{f}_i))$$

and

$$p(\bar{f}_i)\, \bar{M} = \bar{f}_i \bar{M} \qquad \text{and} \qquad (1 - p(\bar{f}_i))\, \bar{M} = (1 - \bar{f}_i)\, \bar{M} \text{ for all } p(\lambda) \text{ in } \mathscr{P}.$$

We shall prove the assertion by means of the $f_i$. Indeed $(f_iM+J^iM)\cap K_i\subseteq J^iM$ (seen by passing to $\bar{R}$) so $\bigcap(f_iM+J^iM)\cap(\bigcap K_i)\subseteq\bigcap J^iM=0$. It remains to show $\bigcap(f_iM+J^iM)+(\bigcap K_i)=M$.

Let $N_i=f_iM+J^iM\subseteq M$. Then $M/J^iM\approx N_i/J^iM\oplus K_i/J^iM$. Hence there is an idempotent $\bar{e}_i$ of $\mathrm{End}_R(M/J^iM)$ satisfying $\bar{e}_i(M/J^iM)=N_i/J^iM$ and $(1-\bar{e}_i)(M/J^iM)=K_i/J^iM$. We have a canonical homomorphism $\pi_i\cdot\mathrm{End}_R M/J^iM\to\mathrm{End}_R M/J^{i-1}M$ since any map $f\colon M/J^iM\to M/J^iM$ satisfies $f(J^{i-1}M/J^iM)\subseteq J^{i-1}M/J^iM$. Now the canonical image of $\bar{e}_i$ in $\mathrm{End}_R(M/J^{i-1}M)$ is $\bar{e}_{i-1}$. By Proposition A the $e_i$ converge to some idempotent $e$ of $\mathrm{End}_R M$ whose image in $\mathrm{End}_R(M/J^iM)$ is $\bar{e}_i$ for each $i$. But then $eM\subseteq\bigcap N_i$, and similarly $(1-e)M\subseteq\bigcap K_i$. Thus $\bigcap N_i+\bigcap K_i=M$ as desired. Q.E.D.

*Note.* Any complete semilocal ring $R$ is semiperfect. Indeed $J^i/J^{i+1}$ is a nilpotent ideal of $R/J^i$ so idempotents can be lifted from $R/J^i$ to $R/J^{i+1}$ and these liftings eventually converge to an idempotent of $R$.

LEMMA 7. *Suppose $R$ is a complete semilocal ring satisfying ACC-$\infty$, and $J$ is f.g. as left ideal. Then $E=\mathrm{End}_R M$ is a local ring for every indecomposable finitely presented module $M$.*

*Proof.* Given $f\colon M\to M$ we want to show either $f$ or $1-f$ is an isomorphism; in view of Proposition 3 it suffices to show $f$ or $1-f$ is onto. By Corollary 4, in the notation of Lemma 6, this is the case if some $f_i$ or $1-f_i$ is onto, so we may assume both $f_i$ and $1-f_i$ are not onto; by "Nakayama's lemma" $(f_iM+J^iM)$ and $(1-f_i)M+J^iM$ are proper submodules of $M$. Since $M$ is indecomposable Lemma 6 yields $\bigcap_{i\in\mathbb{N}}(f_iM+J^iM)=0=\bigcap_{i\in\mathbb{N}}((1-f_i)M+J^iM)$. Thus each $K_i=M=K_{i'}$, implying $f_iM\subseteq J^iM$ and $(1-f_i)M\subseteq J^iM$. But then $M\subseteq J^iM$, contrary to "Nakayama's lemma," so the desired result follows. Q.E.D.

It is standard to use Lemma 7 to build a Krull–Schmidt theory:

THEOREM B. *Suppose $R$ is a complete semilocal Noetherian ring. Then every f.g. $R$-module $M$ has a finite decomposition into a direct sum of indecomposable modules (whose endomorphism rings are local), and this decomposition is unique up to isomorphism and permutation of the summands. Equivalently $\mathrm{End}_R M$ is semiperfect (as well as complete).*

*Proof.* By [8, Proposition 1] we can write $M\approx\bigoplus_{i=1}^{t}M_i$, where each $M_i$ is indecomposable and thus has local endomorphism ring by Lemma 7. Hence we are done by [1, Corollary 12.7 and Corollary 27.7] (since every local ring is semiperfect).

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